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A Lower Bound for the Projection Constant of Π_2

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We denote by Π_n the space of polynomials of degree not greater than n, regarded as a subspace of $C[-1, 1]$. The exact projection constant of Π_2 remains unknown. Chalmers and Metcalf (1) give an upper estimate of 1.2202. In this note, we show that the constant is not less than 1.2158. Perhaps surprisingly, this estimate is obtained while restricting attention to projections defined on a subspace spanned by Π_2 and one extra element.

The following simple (and probably well-known) lemma reduces the problem to one involving only one parameter. Write $f^*(x) = f(-x)$. Call a subspace E of $C[-1, 1]$ "symmetric" if $f^* \in E$ whenever $f \in E$. Similarly, call a linear mapping T "symmetric" if $T(f^*) = (Tf)^*$ for all f. This clearly implies that if f is even (or odd), then so is Tf.

LEMMA 1. *Let E, F be symmetric subspaces of $C[-1, 1]$ with $E \subseteq F$. Given any projection $P_1: F \to E$, there is a symmetric projection $P: F \to E$ with $\|P\| \leqslant \|P_1\|$.*

Proof. Define P_2 by: $P_2 f = (P_1 f^*)^*$. It is elementary that P_2 is also a projection onto E, and that $\|P_2\| = \|P_1\|$. Now let $P = \frac{1}{2}(P_1 + P_2)$.

Now let $u(x) = x\,|x|$, and put $F = \Pi_2 + \operatorname{lin}(u)$. We shall give a lower bound for norms of projections of F onto Π_2. (It might seem more natural to consider projections of Π_3 onto Π_2, but this leads to a worse estimate; we return to this problem below).

By Lemma 1, it is sufficient to consider the projections P_α given by: $(P_\alpha u)(x) = \alpha x$. Note that P_α is in fact the projection defined by interpolation at $-\alpha$, 0, α. Clearly, $\|P_\alpha\| \geqslant \alpha$ for $\alpha > 1$. Further:

LEMMA 2. $\|P_\alpha\| \geqslant 1 + (\alpha^2/4)$ *for* $0 \leqslant \alpha \leqslant 2$.

Proof. Let

$$f(x) = x\,|x| + x^2 - 1 = \begin{cases} -1 & \text{for } x \leqslant 0 \\ 2x^2 - 1 & \text{for } x > 0 \end{cases}.$$

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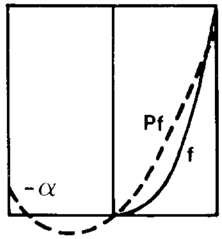

FIGURE 1

Then $\|f\| = 1$, *and*

$$(Pf)(x) = x^2 + \alpha x - 1 = \left(x + \frac{\alpha}{2}\right)^2 - \left(1 + \frac{\alpha^2}{4}\right),$$

so $\|Pf\| = 1 + (\alpha^2/4)$. (See Fig. 1.)

LEMMA 3. $\|P_\alpha\| \geqslant 4 - 3\alpha$ for all α.

Proof. Let g be the function (in F) illustrated. For $x \leqslant 0$,

$$g(x) = 1 - c(x+1)^2 = -cx^2 - 2cx + (1-c)$$

for some c. Since $g(1) = 1$, we have for $x > 0$:

$$g(x) = 3cx^2 - 2cx + (1-c). \qquad \text{(see Fig. 2.)}$$

The requirement that the minimum value is -1 gives $c = \frac{3}{2}$, hence

$$g(x) = 3x\,|x| + \tfrac{3}{2}x^2 - 3x - \tfrac{1}{2}.$$

So we have

$$(P_\alpha g)(x) = \tfrac{3}{2}x^2 - 3(1-\alpha)\,x - \tfrac{1}{2},$$

$$\|P_\alpha g\| \geqslant (P_\alpha g)(-1) = 4 - 3\alpha.$$

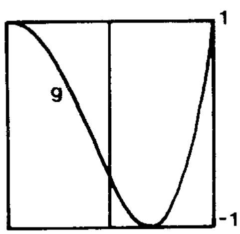

FIGURE 2

We now have two lower estimates for $\|P_\alpha\|$, one increasing with α and the other decreasing. Clearly, it follows that $\|P_\alpha\|$ is never less than the common value where the two functions intersect. This occurs where $\alpha = 4\sqrt{3} - 6 = 0.9282...$, so we have proved:

PROPOSITION 1. *Every projection of F onto Π_2 has norm at least* $1 + 3(7 - 4\sqrt{3}) = 1.21539....$

At the cost of a great increase in complication, this estimate can be raised very slightly. Let $\beta \in (0, 1]$. In Lemma 3, replace g by the function g_β in F whose form for $x \leqslant 0$ is $1 - c(x + \beta)^2$. (So the previous g corresponds to $\beta = 1$.) One finds that

$$\|P_\alpha\| \geqslant 1 + C - D\alpha$$

where

$$C = \frac{\beta + 2}{\beta(\beta + 1)^2}(6\beta - \beta^2 - 1),$$

$$D = 1 + \frac{2}{\beta}.$$

Taking the intersection with $1 + (\alpha^2/4)$ as before, one has that for all α,

$$\|P_\alpha\| \geqslant 1 + C + 2D^2 - 2D\sqrt{C + D^2},$$

with C, D as above. Evaluation shows that the maximum of this expression occurs close to $\beta = 0.955$, where its value is 1.21585 to 5 d.p. (The variation is only about 0.00001 for β between 0.95 and 0.96). So we can state:

PROPOSITION 1′. *Every projection of F onto Π_2 has norm at least* 1.21584.

We now turn to the problem of projections of Π_3 onto Π_2, which is of some independent interest. Again we describe a quick method and then show how the estimate can be improved by taking more trouble. This time, the improvement is more significant.

Let Q_α be the projection that maps the function x^3 to αx; this coincides with interpolation at $-\sqrt{\alpha}$, 0, $\sqrt{\alpha}$.

LEMMA 4. $\|Q_\alpha\| \geqslant 1 + (\alpha^2/4)$.

Proof. The same as Lemma 2, using $f(x) = x^3 + x^2 - 1$.

LEMMA 5. $\|Q_\alpha\| \geqslant 1 + \frac{27}{16}(1 - \alpha)$.

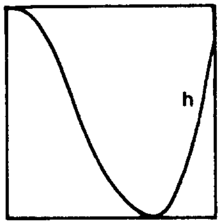

FIGURE 3

Proof. Let $h(x) = 1 + c(x+1)^2(x-1)$, with c chosen so that the minimum value (which occurs at $x = \frac{1}{3}$) is -1. This gives $c = \frac{27}{16}$, so

$$(Q_\alpha h)(x) = 1 + \tfrac{27}{16}(x^2 - (1-\alpha)\,x - 1),$$

$$(Q_\alpha h)(-1) = 1 + \tfrac{27}{16}(1-\alpha).$$

(See Fig. 3.)

PROPOSITION 2. *Every projection of Π_3 onto Π_2 has norm at least* 1.1954.

Proof. Equating the expressions in Lemmas 4 and 5, we find $\alpha = \frac{1}{8}(\sqrt{27 \times 43} - 27) = 0.88418\ldots$, which gives the estimate stated.

Our improved estimate is obtained by varying the functions used in both Lemma 4 and Lemma 5, as follows.

LEMMA 4′. *For* $0 \leqslant c \leqslant 1$,

$$\|Q_\alpha\| \geqslant \frac{1+2c}{(1+c)^2}\left[1 + \frac{1}{4}\left(\frac{\alpha + c(2+c)}{1+2c}\right)^2\right] = F(c, \alpha).$$

Proof. Let

$$f_c(x) = \frac{(x+c)^2(x+1)}{(1+c)^2} - 1.$$

Then $\|f_c\| = 1$ and

$$(1+c)^2(Q_\alpha f_c)(x) = (1+2c)(x^2-1) + (\alpha + 2c + c^2)\,x,$$

from which the statement follows. (See Fig. 4.)

LEMMA 5′. *For* $0 \leqslant \beta \leqslant 1$,

$$\|Q_\alpha\| \geqslant 1 + \frac{27}{2(1+\beta)^3}(4\beta - 2\beta^2 - 1 - \alpha).$$

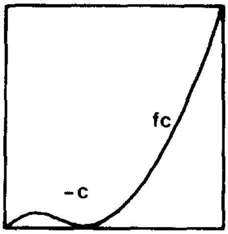

FIGURE 4

For given α, this is maximized by $\beta = 3 - \sqrt{(11 - 3\alpha)/2}$. (Note the resulting expression by $G(\alpha)$.)

Proof. Let $h_\beta(x) = 1 + c(x+\beta)^2(x-1)$, with $c = (27/2(1+\beta)^3)$ so that the minimum value is -1. Then $(Q_\alpha h_\beta)(-1)$ is the expression stated.

PROPOSITION 2′. *The estimate in Propositon 2 can be replaced by* 1.199.

Proof. By trial and error, we find

$$F(0.05, 0.8857) = 1.1991\ldots, \qquad G(0.8857) = 1.1992\ldots.$$

REFERENCE

1. B. L. CHALMERS AND F. T. METCALF, There exists a minimal projection from C onto V_{n+1} which is either singular or has finite carrier. Preprint.